

FACTOR SETS AND DIFFERENTIALS ON ABELIAN VARIETIES

BY

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The main results of this paper are stated in 2.8 and 3.7; the notations and language are those used by the author in his previous work; numbers in brackets refer to the bibliography at the end of the paper. Although a few results of §§1 and 2 are valid for any characteristic, this paper intends to treat only the case of characteristic zero. The case of positive characteristic is treated in a forthcoming paper, since it needs an analysis of the derivations of higher order, and is connected to the rather surprising fact that an abelian variety over a field of positive characteristic may very well possess exact differentials of the first kind. §§1 and 2 contain the proofs of those properties of the differentials of the second kind which are needed in §3; these properties are familiar in classical algebraic geometry, but their algebraic proofs are new. While our main interest, in §§1 and 2, rests with the differentials of the second kind, certain properties of the differentials of the first kind are also found, usually as special cases; the algebraic proofs of most of these properties are not new, and can be found also in [S. Koizumi, *On the differential forms of the first kind on algebraic varieties*, J. Math. Soc. Japan vol. 1 (1949) p. 273] and [S. Nakano, *On invariant differential forms on group varieties*, *ibid.* vol. 2 (1951) p. 216].

1. **The differentials of the first and second kind.** Let V be an irreducible variety over the field k ; let W be an irreducible subvariety of V , and set $Q = Q(W/V)$, $\mathfrak{P} = \mathfrak{P}(W/V)$; let D be a derivation on V (see §5 of [4]). We shall say that D is *regular at W* if $Dx \in Q$ when $x \in Q$, and $Dx \in \mathfrak{P}$ when $x \in \mathfrak{P}$. If, in addition, $Dx \in \mathfrak{P}$ when $x \in Q$, we shall say that D *has a zero at W* . Let π be the homomorphic mapping of Q into $k(W)$ whose kernel is $\mathfrak{P}$; if D is regular at W , for $x \in Q$ we see that $\pi(Dx)$ depends only on πx , and from this follows the existence of a derivation πD on W such that $(\pi D)(\pi x) = \pi(Dx)$ for $x \in Q$; πD is called the derivation *induced by D on W* ; clearly, $\pi D = 0$ if and only if D has a zero at W .

1.1. **THEOREM.** *Let V be an n -dimensional irreducible variety of inseparability 1 over the field k ; let W be an m -dimensional irreducible subvariety of V , simple on V , of inseparability 1. Set $Q = Q(W/V)$, $\mathfrak{P} = \mathfrak{P}(W/V)$, and let π be the homomorphic mapping of Q onto $k(W)$ whose kernel is $\mathfrak{P}$. Let $x_1, \dots, x_m$*

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be elements of Q such that $k(x_1, \dots, x_m) \subseteq Q$, and that $k(W)$ be an algebraic separable extension of $k(\pi x_1, \dots, \pi x_m)$; let $\{x_{m+1}, \dots, x_n\}$ be a regular set of parameters of Q . There exist derivations $D_1, \dots, D_n$ on V such that $D_i x_j = \delta_{ij}$ (Kronecker's symbol) for $i, j = 1, \dots, n$; these form an independent $k(V)$ -basis for the derivations on V . The derivations on V regular at W are all and only those of the form $\sum_{i=1}^n a_i D_i$, with $a_i \in Q$ for $i \leq m$, and $a_i \in \mathfrak{P}$ for $i > m$; among these, those with a zero at W are all and only those for which $a_i \in \mathfrak{P}$ for $i \leq m$. The derivations πD_i ($i = 1, \dots, m$) form an independent $k(W)$ -basis for the derivations on W .

Proof. Let F be the projective space over k whose n.h.g.p. (nonhomogeneous general point) is $\{x_1, \dots, x_n\}$; let $\{y_1, \dots, y_r\}$ be a n.h.g.p. of V , for which W is at finite distance, and let y_0 be an element of Q . Let V' be the model of $k(V)$ whose n.h.g.p. is $\{y_0, \dots, y_r\}$; then W corresponds on V' to an irreducible variety W' such that $Q(W'/V') = Q$. Let ϕ be the rational mapping of V' onto F generated by the embedding of $k(F)$ into $k(V')$; let $\{Y_0, \dots, Y_r\}$ be the n.h.g.p. of the ambient space of $\phi[F]$, and let $\{f_1(Y, x), f_2(Y, x), \dots\}$ be a basis of the ideal related to ϕ in $k[Y, x]$. The Corollary to Theorem 5.6 of [2] (modified to apply to nonhomogeneous co-ordinates) implies then that the jacobian $J(f(Y, x); Y)$ acquires the rank $r+1$ when $\{Y\}$ is replaced by $\{\pi y\}$ and $\{x\}$ by $\{\pi x\}$, hence also when $\{Y\}$ is replaced by $\{y\}$ and $\{x\}$ by $\{x\}$; this, in particular, indicates that $\text{ins}(k(V):k(F)) = 1$, so that $D_1, \dots, D_n$ exist and form an independent $k(V)$ -basis for the derivations on V . From $f_i(y, x) = 0$ follows $D_j f_i(y, x) = 0$, or

$$[\partial f_i(Y, x)/\partial x_j]_{Y=y} + \sum_h [\partial f_i(Y, x)/\partial Y_h]_{Y=y} D_j y_h = 0.$$

If these are considered as equations in the unknowns $D_j y_0, \dots, D_j y_r$, the previous remark on the rank of $J(f; Y)$ shows that $D_j y_h \in Q$, in particular $D_j y_0 \in Q$, valid for any $y_0 \in Q$. If $y_0 \in \mathfrak{P}$, then $y_0 = \sum_{i=m+1}^n a_i x_i$, $a_i \in Q$, so that, for $j \leq m$, $D_j y_0 = \sum_{i=m+1}^n x_i D_j a_i \in \mathfrak{P}$; hence $D_1, \dots, D_m$ are regular at W . Moreover, $b D_j$ ($j = 1, \dots, n$) certainly has a zero at W if $b \in \mathfrak{P}$.

Now, let $\sum_{i=1}^n a_i D_i$ be regular at W ; then $\sum_i a_i D_i x_j = a_j$ must be in Q if $j \leq m$, and in $\mathfrak{P}$ if $j > m$; finally, if $\sum_i a_i D_i$ also has a zero at W , it must necessarily be $a_i \in \mathfrak{P}$ for $i = 1, \dots, n$, as announced. The derivations πD_i ($i = 1, \dots, m$) exist and are independent, since $(\pi D_i)(\pi x_j) = \delta_{ij}$, Q.E.D.

The derivations $D_1, \dots, D_n$ of 1.1, in this order, are said to form a *canonical W -basis* for the derivations on V , related to the set $\{x_1, \dots, x_n\}$; this set, in turn, is called a *set of uniformizing parameters at W on V* .

Let V be an irreducible variety over k ; a *differential of $k(V)$ over k* , or *on V* , is a linear mapping $\omega: D \rightarrow \omega D$ of the $k(V)$ -module of the derivations on V into $k(V)$; linear in the sense that if D, D' are derivations, and $x \in k(V)$, then $\omega D \in k(V)$, $\omega(D + D') = \omega D + \omega D'$, $\omega(xD) = x\omega D$. If ω, ω' are differentials on V , and $x \in k(V)$, the differentials $\omega + \omega'$, $x\omega = \omega x$ are defined by $(\omega + \omega')D$

$=\omega D+\omega'D$, $(x\omega)D=x(\omega D)$ for any derivation D on V . The differentials on V thus form a $k(V)$ -module (dual to the $k(V)$ -module of the derivations on V). If $\text{ins } V=1$, such module has order $n=\dim V$. If $x\in k(V)$, dx (or, if necessary, d_Vx) shall denote the differential on V such that $(dx)D=Dx$ for any derivation D on V . A differential of the type dx is called an *exact* differential.

Let W be an irreducible subvariety of V ; set $Q=Q(W/V)$, $\mathfrak{P}=\mathfrak{P}(W/V)$, and let ω be a differential on V . We say that ω has a *pole at* W if either $\omega D\notin Q$ for some derivation D on V , regular at W , or $\omega D\notin \mathfrak{P}$ for some derivation D on V which has a zero at W ; if ω does not have a pole at W , it is said to be *regular at* W . If this is the case, and if π denotes the homomorphic mapping of Q onto $k(W)$ whose kernel is $\mathfrak{P}$, then, for any derivation D on V , regular at W , $\pi(\omega D)$ depends only on πD , and therefore there exists at least one differential ω' on W such that $\omega'(\pi D)=\pi(\omega D)$ for any such D . If the πD are all the derivations on W , as it is under the assumptions of 1.1, ω' is unique, and is then denoted by $\pi\omega$ or $\omega\cap W$ or $W\cap\omega$, and called the differential on W *induced by* ω ; ω induces 0 on W , or *has a zero at* W , if $\pi\omega=0$, or if $\omega D\in \mathfrak{P}$ for each derivation D on V , regular at W .

1.2. THEOREM. *Let k , V , W , $\{x\}$, Q , $\mathfrak{P}$, π be as in 1.1; then the differentials on V regular at W are all and only those of the form $\sum_{i=1}^n a_i dx_i$, with $a_i\in Q$ ($i=1, \dots, n$); and among these, those with a zero at W are all and only those for which $a_i\in \mathfrak{P}$ ($i=1, \dots, m$). The differentials $W\cap dx_i$ ($i=1, \dots, m$) form an independent $k(W)$ -basis for the differentials on W . If $x\in Q$, dx is regular at W , and $d\pi x=W\cap dx$; if $x\in \mathfrak{P}$, dx has a zero at W .*

Proof. For $i=1, \dots, n$, dx_i is regular at W ; in fact, if $D_1, \dots, D_n$ have the same meaning as in 1.1, and if $D=\sum_i a_i D_i$ is regular at W , we have $(dx_i)D=Dx_i=a_i$, and this, by 1.1, belongs to Q , and also to $\mathfrak{P}$ if D has a zero at W . If $\omega=\sum_i b_i dx_i$ is regular at W , then $\omega D=\sum_i b_i a_i$ must belong to Q if $a_1, \dots, a_m\in Q$ and $a_{m+1}, \dots, a_n\in \mathfrak{P}$, and to $\mathfrak{P}$ if also $a_1, \dots, a_m\in \mathfrak{P}$. This is true if and only if $b_1, \dots, b_n\in Q$; and ω has a zero at W if and only if $\sum_i b_i a_i\in \mathfrak{P}$ for $a_1, \dots, a_m\in Q$, $a_{m+1}, \dots, a_n\in \mathfrak{P}$; this is true if and only if $b_1, \dots, b_m\in \mathfrak{P}$ and $b_{m+1}, \dots, b_n\in Q$. The differentials $W\cap dx_i$ ($i=1, \dots, m$) exist, and are independent over $k(W)$; hence they form a $k(W)$ -basis for the differentials on W .

If $x\in Q$, then $dx=\sum_i (D_i x)dx_i$, and this is regular at W because $D_i x\in Q$ for $i=1, \dots, n$; if $x\in \mathfrak{P}$, then $x=\sum_{j=m+1}^n y_j x_j$, with $y_j\in Q$; hence $dx=\sum_{j=m+1}^n y_j dx_j+x_j dy_j$; since each dx_j and each $x_j dy_j$ has a zero at W , the same is true of dx . The statement $W\cap dx=d\pi x$ is obvious, Q.E.D.

The differentials $dx_1, \dots, dx_n$ of 1.2, ordered so that those with a zero at W are the last $n-m$, are said to form a *canonical W -basis* for the differentials on V , *related to* the set $\{x_1, \dots, x_n\}$.

1.3. LEMMA. *Let V , W be as in 1.1, and let ω be a differential on V which*

has a pole at W ; then ω has a pole at some $(n-1)$ -dimensional irreducible subvariety of V containing W .

Proof. Set $Q = Q(W/V)$, $\mathfrak{P} = \mathfrak{P}(W/V)$. If ω has a pole at W , then either $\omega D \notin Q$ for some derivation D on V , regular at W , or $\omega D \notin \mathfrak{P}$ for some derivation D on V with a zero at W . In the first case, set $\omega D = qp^{-1}$, where $q \in Q$, $p \in \mathfrak{P}$, and p, q have no common factors (here one should remember that every minimal prime ideal of Q is principal; see [8]). If $D' = pD$, D' has a zero at W ; let U be the center on V of a prime divisor v of V such that $Q \subseteq R_v$ and that $v(p) > 0$; then $D'x \in pQ \subseteq \mathfrak{P}_v$ for $x \in Q$, so that if $a = xy^{-1} \in R_v$, with $x \in Q$, $y \in Q - pQ$, we have $D'a = y^{-1}D'x - xy^{-2}D'y \in \mathfrak{P}_v$. Hence D' has a zero at U ; but $\omega D' = p\omega D = q \notin \mathfrak{P}_v$, which proves that ω has a pole at U . In the second case, i.e. when $\omega D \notin \mathfrak{P}$ and D has a zero at W , we may assume $\omega D = q$ to be a unit of Q , otherwise this case would fall within the first case. If $\{D_1, \dots, D_n\}$ is a canonical W -basis for the derivations on V , we have $D = \sum_i a_i D_i$, $a_i \in \mathfrak{P}$, and $q = \omega D = \sum_i a_i \omega D_i$. Unless ω is such that the first case applies, we must have $\omega D_i \in Q$ for $i \leq m$, and therefore $\sum_{i=m+1}^n a_i \omega D_i$ is a unit of Q , while each $a_i \omega D_i$ belongs to Q . As a consequence, $a_i \omega D_i$ is a unit of Q for some $i > m$, say $i = n$. Now, let U, v be related to a_n as they were before to p ; if $D' = a_n D_n$, we have $D'x \in a_n Q \subseteq \mathfrak{P}_v$ for $x \in Q$, and therefore also for $x \in R_v$; hence D' has a zero at U , but $\omega D' \notin \mathfrak{P}_v$. This proves that ω has a pole at U , Q.E.D.

The sentence " U is a pole of ω " shall mean, from now on, that ω has a pole at U , and that U has dimension $n-1$, if n is the dimension of V .

1.4. THEOREM. *Let V be an n -dimensional irreducible variety over k ; let W be an irreducible proper subvariety of V , simple on V , and of inseparability 1; let ω be a differential on V . Then ω has a pole at W if and only if W is a subvariety of a pole of ω . If $x \in k(V)$, and V has no $(n-1)$ -dimensional singularity, and no $(n-1)$ -dimensional irreducible subvariety of inseparability > 1 , each pole of dx is a pole of x ; under the same assumptions, ω has finitely many poles. If, in addition, k has characteristic 0, each pole of x is a pole of dx .*

Proof. If ω has a pole at W , then a pole of ω contains W by 1.3. If $W \subseteq U$ and U is a pole of ω , let $\{dx_1, \dots, dx_n\}$ be a canonical W -basis for the differentials on V , and write $\omega = \sum_i a_i dx_i$; should ω not have a pole at W , we would have $a_i \in Q(W/V) \subseteq Q(U/V)$ for each i by 1.2. Since, by 1.2, dx_i is regular at U , the same would be true of ω , a contradiction. Hence ω has a pole at W . If U is now any $(n-1)$ -dimensional irreducible subvariety of V , simple on V , and if $\text{ins } U = 1$, from $x \in Q(U/V)$ follows that dx is regular at U by 1.2; hence each pole of dx , under the additional conditions mentioned in the statement, is a pole of x . Therefore the poles of ω are to be found among the poles of $x_1, \dots, x_n, a_1, \dots, a_n$, and are finite in number. If k has characteristic 0, and U is a pole of x , we can write $x = ut^r$, where u is a unit of $Q(U/V)$, t generates $\mathfrak{P}(U/V)$, and $r < 0$. Then $dx = rut^{r-1}dt + t^r du$; let

$\{t_1, \dots, t_n\}$ be a set of uniformizing parameters at U on V , such that $t_n = t$, and let $\{D_1, \dots, D_n\}$ be the canonical U -basis for the derivations on V which is related to $\{t\}$. Then $(dx)(tD_n) = rut^r + t^{r+1}D_nu$; since $tD_nu \in \mathfrak{P}(U/V)$, it follows that $t^{r+1}D_nu \in t^{r+1}Q(U/V)$, while $rut^r \notin t^{r+1}Q(U/V)$; hence $(dx)(tD_n) \notin Q(U/V)$, and therefore U is a pole of dx , since tD_n has a zero at U by 1.1, Q.E.D.

1.5. THEOREM. Let U, W, V be irreducible varieties over k , such that (1) $U \subset W \subset V$, (2) U is simple on W and V , and (3) $\text{ins } U = 1$; let ω be a differential on V , regular at U . Then ω is regular at W , and $\omega \cap U = (\omega \cap W) \cap U$.

Proof. By conditions (2) and (3), if the dimensions of W, V are m, n respectively, it is possible to find a set of uniformizing parameters $\{x_1, \dots, x_n\}$ at U on V which is also a set of uniformizing parameters at W on V , and such that if π is the homomorphic mapping of $Q(W/V)$ onto $k(W)$ whose kernel is $\mathfrak{P}(W/V)$, $\{\pi x_1, \dots, \pi x_m\}$ is a set of uniformizing parameters at U on W . If ω is regular at U , we have $\omega = \sum_{i=1}^n a_i dx_i$, with $a_i \in Q(U/V)$, by 1.2; hence $a_i \in Q(W/V)$, and ω is regular at W . But then, by 1.2, $\omega \cap W = \sum_{i=1}^m (\pi a_i) d\pi x_i$; now, $\pi a_i \in Q(U/W)$, so that $\omega \cap W$ is regular at U by 1.2, and the relation $(\omega \cap W) \cap U = \omega \cap U$ follows easily, Q.E.D.

A differential ω on V is said to be of the first kind (on V) if it is regular at each irreducible subvariety of V ; by 1.4, if V has no singularity, and no irreducible subvariety with inseparability > 1 , a differential on V is of the first kind if and only if it has no poles (of dimension $n-1$ if $n = \dim V$). Two differentials ω, ω' on V are said to be equivalent, in symbols $\omega \sim \omega'$, if $\omega' - \omega$ is exact. A differential ω on V is said to be of the second kind (on V) if for each irreducible proper subvariety W of V there exists a differential $\omega' \sim \omega$ which is regular at W . From 1.5 we obtain:

1.6. COROLLARY. Let W be an irreducible subvariety of the irreducible variety V over the perfect field k , and assume each point of W to be simple on W and V . Let ω be a differential on V , regular at W ; if ω is of the first (second) kind, so is $W \cap \omega$.

Two derivations D, Δ on V are said to commute if $D\Delta x = \Delta D x$ for each $x \in k(V)$; a differential ω on V is said to be closed if $D(\omega\Delta) = \Delta(\omega D)$ for each pair $\{D, \Delta\}$ of derivations on V which commute. Any exact differential on V is closed; and any differential on V is closed if $\dim V = 1$, as will clearly appear from the following result:

1.7. LEMMA. Let V be an n -dimensional irreducible variety over k , of inseparability 1. A differential ω on V is closed if and only if $D_i(\omega D_j) = D_j(\omega D_i)$ for $i, j = 1, \dots, n$, where $\{D_1, \dots, D_n\}$ is a $k(V)$ -basis for the derivations on V , consisting of mutually commutative derivations.

Proof. If $D = \sum_i a_i D_i$, and $\Delta = \sum_i b_i D_i$ are commutative derivations on

V , with $a_i, b_i \in k(V)$, we have $Db_i = \Delta a_i$, and $D(\omega\Delta) = \sum_i (Db_i)\omega D_i + b_i D(\omega D_i) = \sum_i (\Delta a_i)\omega D_i + a_i \Delta(\omega D_i) = \Delta(\omega D)$, Q.E.D.

It is evident that if ω is closed, also $W \cap \omega$ is closed if it exists.

Let V be an absolutely irreducible variety over k , and let K be an extension of k . If D is a derivation on V , there is a unique derivation D_K on V_K such that $D_K x = D x$ for any $x \in k(V) \subseteq K(V_K)$. Likewise, if ω is a differential on V , there is a unique differential ω_K on V_K such that $\omega_K D_K = \omega D$ for any derivation D on V ; in particular, $d_{V_K} x = (d_V x)_K$ for $x \in k(V)$. A somewhat different situation arises in considering two irreducible varieties V, V' such that $k(V) \subseteq k(V')$ and $\text{ins } V = \text{ins } (k(V') : k(V)) = 1$; in this case, for $x \in k(V)$, we have $d_{V'} x = 0$ if and only if $d_V x = 0$; if ω is a differential on V , and $\omega = \sum_i a_i d_V x_i$, where $a_i, x_i \in k(V)$, then $\sum_i a_i d_{V'} x_i$ gives a differential on V' , which does not depend on the choice of the x 's, and which is 0 if and only if $\omega = 0$. Such differential is called the *extension of ω on V'* . A particular case arises when $V' = V \times W$, where W is another absolutely irreducible variety over k (V being also absolutely irreducible in this case); then the extension of ω on V' will be denoted by $\omega \times W$ or $W \times \omega$; there is an obvious one-to-one correspondence $\omega \times W \rightarrow \omega_K$, for $K = k(W)$. Also, $\omega \times W$ or ω_K are closed if and only if ω is closed.

1.8. LEMMA. *Let V be an irreducible variety without singularities over the algebraically closed field k ; let K be an extension of k , and let ω be a differential on V ; then ω_K is of the first (second) kind if and only if ω is of the first (second) kind.*

Proof. Set $n = \dim V$, $d = d_V$, $d' = d_{V_K}$; assume ω to be of the second kind' and let U be a nonempty irreducible subvariety of V_K ; there exists a point $P \in V$ such that $Q(P/V) \subseteq Q(U/V_K)$, and there exists an $f \in k(V)$ such that $\omega - df$ is regular at P ; in particular, $f = 0$ if ω is of the first kind. Thus, $\omega - df = \sum_i a_i dx_i$, where $a_i, x_i \in Q(P/V)$, by 1.2; but then, again by 1.2, $\omega_K - d'f$ is regular at U , so that ω_K is of the second kind, or of the first kind if $f = 0$.

Now, assume ω_K to be of the second kind; let U be a nonempty irreducible subvariety of V , and let $f \in K(V_K)$ be such that $\omega_K - d'f$ is regular at U_K ; here, $f = 0$ if ω_K is of the first kind. Let v be any place of K over k , whose extension w to $K(V_K)$ over $k(V)$ is such that $w(f) \geq 0$, and denote by $\bar{}$ the homomorphic image of an element of $R_w \bmod \mathfrak{P}_w$. Write $\omega_K - d'f = \sum_i a_i d'x_i$, with $a_i \in Q(U_K/V_K)$, $\{x_1, \dots, x_n\}$ being a set of uniformizing parameters at U on V . The poles of $\omega_K - d'f$ are among the poles of the a_i and the x_i , by 1.4, and none of them contains U_K , again by 1.4; hence, after a suitable choice of v , none of the poles of any $\bar{a}_i$ or any $\bar{x}_i$ contains U , and therefore $\omega - d\bar{f}$ is regular at U , Q.E.D.

If V is an irreducible variety over the field k , we shall denote by $\mathfrak{D}_1(V)$, $\mathfrak{D}_2(V)$, $\mathfrak{D}_e(V)$ respectively the k -modules of the closed differentials of the first kind on V , of the closed differentials of the second kind on V , and of the

exact differentials on V ; clearly, $\mathfrak{D}_1(V)$ and $\mathfrak{D}_e(V)$ are sub-modules of $\mathfrak{D}_2(V)$, their intersection being 0 if k has characteristic 0, by 1.4. If V is absolutely irreducible, we shall say that an independent k -basis $\{\omega_1, \omega_2, \dots\}$ of $\mathfrak{D}_1(V)$ is an *absolute basis*, if $\{(\omega_1)_K, (\omega_2)_K, \dots\}$ is a K -basis of $\mathfrak{D}_1(V_K)$, for each algebraically closed extension K of k . And a set $\{\omega_1, \omega_2, \dots\}$ of elements of $\mathfrak{D}_2(V)$, whose images mod $\mathfrak{D}_e(V)$ form an independent k -basis of $\mathfrak{D}_2(V)/\mathfrak{D}_e(V)$, is said to *represent an absolute basis*, if the images of $(\omega_1)_K, (\omega_2)_K, \dots$, mod $\mathfrak{D}_e(V_K)$, form a K -basis of $\mathfrak{D}_2(V_K)/\mathfrak{D}_e(V_K)$, for any algebraically closed extension K of k .

If $\{\omega_1, \omega_2, \dots\}$ is an absolute basis of $\mathfrak{D}_1(V)$, or, respectively, a representative of an absolute basis of $\mathfrak{D}_2(V)/\mathfrak{D}_e(V)$, then $\{(\omega_1)_K, (\omega_2)_K, \dots\}$ is a K -basis of $\mathfrak{D}_1(V_K)$, or, respectively, represents a K -basis of $\mathfrak{D}_2(V_K)/\mathfrak{D}_e(V_K)$, for any extension K of k . We shall give the proof of this statement in the second, less obvious, case. Let K be an extension of k , and let $\omega \in \mathfrak{D}_2(V_K)$; then, for a suitable finite extension H of K , we can find an $x \in H(V_H)$, and elements $a_1, a_2, \dots \in H$, such that $\omega_H - d_{V_H}x = \sum_i a_i(\omega_i)_H$. Let $\{b_1 = 1, b_2, \dots, b_r\}$ be an independent K -basis of H ; then $\{b\}$ is also an independent $K(V_K)$ -basis of $H(V_H)$. Write $x = \sum_j x_j b_j$, $x_j \in K(V_K)$, and $a_i = \sum_j a_{ij} b_j$, $a_{ij} \in K$. Then $\omega_H - d_{V_H}x_1 - \sum_i a_{i1}(\omega_i)_H = \sum_{j=2}^r (d_{V_H}x_j + \sum_i a_{ij}(\omega_i)_H)b_j$. If D is any derivation on V , this gives $\omega D_K - D_K x_1 - \sum_i a_{i1}\omega_i D = \sum_{j=2}^r (D_K x_j + \sum_i a_{ij}\omega_i D)b_j$; since the left member, and each $D_K x_j + \sum_i a_{ij}\omega_i D$, for $j > 1$, belong to $K(V_K)$, we conclude that $\omega D_K - D_K x_1 = \sum_i a_{i1}\omega_i D$, or that $\omega - d_{V_K}x_1 = \sum_i a_{i1}(\omega_i)_K$, as claimed.

1.9. THEOREM. *Let V_1, V_2 be irreducible varieties without singular points over the algebraically closed field k , and set $V = V_1 \times V_2$. For a given j ($= 1, 2$), assume $\omega_i \in \mathfrak{D}_j(V_i)$ ($i = 1, 2$); then $\omega_1 \times V_2 + \omega_2 \times V_1 \in \mathfrak{D}_j(V)$. If ω_1, ω_2 are differentials on V_1, V_2 respectively, and $\omega = \omega_1 \times V_2 + \omega_2 \times V_1 \in \mathfrak{D}_j(V)$ ($j = 1$ or 2), then $\omega_i \in \mathfrak{D}_j(V_i)$ for $i = 1, 2$; also, ω_1, ω_2 are uniquely determined by ω . If $\mathfrak{D}_1(V_i)$ has an absolute basis for $i = 1, 2$, and k has characteristic zero, then $\mathfrak{D}_1(V)$ has an absolute basis, and each element of $\mathfrak{D}_1(V)$ can be expressed in the form $\omega_1 \times V_2 + \omega_2 \times V_1$, with $\omega_i \in \mathfrak{D}_1(V_i)$. If there exists a representative of an absolute basis of $\mathfrak{D}_2(V_i)/\mathfrak{D}_e(V_i)$ for $i = 1, 2$, and k has characteristic zero, then there exists a representative of an absolute basis of $\mathfrak{D}_2(V)/\mathfrak{D}_e(V)$, and each element of $\mathfrak{D}_2(V)$ is equivalent ($\sim$) to an element of the form $\omega_1 \times V_2 + \omega_2 \times V_1$, with $\omega_i \in \mathfrak{D}_2(V_i)$ ($i = 1, 2$), uniquely determined but for equivalence.*

Proof. PART 1. Set $n_i = \dim V_i$; if $\omega = \omega_1 \times V_2 + \omega_2 \times V_1$, any pole of ω is of type $U_1 \times V_2$ or $U_2 \times V_1$, with U_i a pole of ω_i . Hence, by 1.4, $\omega \in \mathfrak{D}_i(V)$ ($i = 1, 2$) if and only if $\omega_j \in \mathfrak{D}_i(V_j)$ for $j = 1, 2$.

PART 2. Assume k to have characteristic zero, and let $\{\alpha_{i1}, \alpha_{i2}, \dots\}$ represent an absolute basis of $\mathfrak{D}_2(V_i)/\mathfrak{D}_e(V_i)$ ($i = 1, 2$). Let $\omega \in \mathfrak{D}_2(V)$, and write $\omega = \sum_i a_i dx_i + \sum_j b_j dy_j$, where $a_i, b_j \in k(V)$, and $\{x_1, \dots, x_{n_1}\}, \{y_1, \dots, y_{n_2}\}$ are algebraically independent (over k) elements of $k(V_1)$,

$k(V_2)$ respectively, such that $k(V_1)$, $k(V_2)$ are finite separable extensions of, respectively, $k(x)$ and $k(y)$. Set $K = k(V_1)$, $H = k(V_2)$, $d^* = d_{(V_1)_H}$, $\omega^* = \sum_i a_i d^* x_i$. Then ω^* is a closed differential on $(V_1)_H$; if W is an irreducible subvariety of $(V_1)_H$ at which ω^* has a pole, set $W' = D_W V_2$, so that W' is a subvariety of V such that $Q(W'/V) = Q(W/(V_1)_H)$. There exists a $z \in k(V)$ such that $\omega - dz$ is regular at W' ; but then $\omega^* - d^* z$ is regular at W . This proves that $\omega^* \in \mathfrak{D}_2((V_1)_H)$; as a consequence, there exists an $x \in k(V)$ such that $\omega^* - d^* x$ is a linear combination, with coefficients in H , of, say, $(\alpha_{11})_H, \dots, (\alpha_{1r})_H$. Set $\alpha_{1s} = \sum_j t_{sj} d_1 x_j$, $t_{sj} \in K$; then there are elements $v_1, \dots, v_r$ of H such that $\omega - dx = \sum_{ij} v_i t_{ij} dx_j + \sum_j w_j dy_j$, with $w_j \in k(V)$. This is a closed differential; if D_i, Δ_j are the derivations on V such that $D_i x_j = \delta_{ij}$, $D_i y_j = 0$, $\Delta_i x_j = 0$, $\Delta_i y_j = \delta_{ij}$, we must have $D_i w_j = \sum_h t_{hi} \Delta_j v_h$. This implies that the poles of $d^* w_j$ are all of the type U_H , with U an $(n_1 - 1)$ -dimensional irreducible subvariety of V ; consequently, by 1.4, the poles of w_j on $(V_1)_H$ are of the same type, and therefore $w_j \in K \times H$ (direct product over k). Write then $w_j = \sum_s c_{js} e_{js}$, with $c_{js} \in K$, $e_{js} \in H$, and $e_{j1}, e_{j2}, \dots$ linearly independent over k , hence over K . Then $\sum_h (\Delta_j v_h) t_{hi} = D_i w_j = \sum_s e_{js} D_i c_{js}$; this shows, by (99) of [1], that there are elements $\beta_{jhs} \in k$ such that $\Delta_j v_h = \sum_s \beta_{jhs} e_{js}$, and that $D_i c_{js} = \sum_h \beta_{jhs} t_{hi}$, so that $d_1 c_{js} = \sum_{hi} \beta_{jhs} t_{hi} d_1 x_i = \sum_h \beta_{jhs} \alpha_{1h}$. Since $\alpha_{11}, \dots, \alpha_{1r}$ are independent over $k \bmod \mathfrak{D}_e(V_1)$, we conclude that $\beta_{jhs} = 0$, so that $\Delta_j v_h = 0$, $v_h \in k$; also, $D_i c_{js} = 0$, $c_{js} \in k$, and $w_j \in H$. Thus $\omega - dx = \omega_1 \times V_2 + \omega_2 \times V_1$, where $\omega_1 = \sum_i v_i \alpha_i$ is a differential on V_1 , and $\omega_2 = \sum_j w_j d_2 y_j$ is a differential on V_2 . By Part 1 of this proof, ω_1 and ω_2 are closed and of the second kind; they are unique but for equivalence ($\sim$), because if $\tilde{\omega} = \omega_1 \times V_2 + \omega_2 \times V_1 \sim 0$, then, for a generic $P_2 \in V_2$, also $\omega_1 \times P_2 = \omega \cap (V_1 \times P_2) \sim 0$.

PART 3. The statement concerning the case in which ω is of the first kind is proved in the same manner (actually simpler) used in Part 2.

PART 4. The statement concerning the existence of absolute bases is a direct consequence of Parts 3 and 4, Q.E.D.

2. **Application to abelian varieties.** Let G be a group-variety, without singular points outside the degeneration locus F , over the field k . If P is a rational point of G , σ_P and τ_P will have the usual meanings (see [3] and [4]). If D is a derivation on G , we shall denote by $\sigma_P D$ the derivation on G such that $(\sigma_P D)(\sigma_P x) = \sigma_P(Dx)$ for any $x \in k(G)$; $\tau_P D$ is similarly defined. When k is algebraically closed, and according to §5 of [4], D is left (respectively right) invariant if and only if $\sigma_P D = D$ (respectively $\tau_P D = D$) for any $P \in G - F$. Let ω be a differential on G ; we shall denote by $\sigma_P \omega$ the differential on G such that $(\sigma_P \omega)(\sigma_P D) = \sigma_P(\omega D)$ for any derivation D on G ; in particular, $\sigma_P dx = d\sigma_P x$ for $x \in k(G)$; $\tau_P \omega$ is similarly defined. If k is algebraically closed, ω is said to be *left (right) invariant* if $\sigma_P \omega = \omega$ (if $\tau_P \omega = \omega$) for any $P \in G - F$; if both relations are fulfilled, ω is said to be *invariant*. Under the same conditions, ω is said to be *left (right) semi-invariant* if for each $P \in G - F$ there exists an

$x_P \in k(G)$ such that $\sigma_P \omega - \omega = dx_P$ (respectively $\tau_P \omega - \omega = dx_P$); if both relations are fulfilled (not necessarily for the same x_P), ω is *semi-invariant*.

2.1. THEOREM. *Let A be an n -dimensional nonsingular abelian variety over the algebraically closed field k . Then the invariant differentials on A are all elements of $\mathfrak{D}_1(A)$, and form a k -module of order n . Also, each differential of the first kind on A is invariant, hence closed, and $\mathfrak{D}_1(A)$ has an absolute basis.*

Proof. Let $D_1, \dots, D_n$ be the invariant derivations on A defined in §5 of [4], and let ω_i be the differential on A defined by $\omega_i D_j = \delta_{ij}$. Then $\omega_1, \dots, \omega_n$ are invariant, and linearly independent over k (also over $k(A)$). If ω is an invariant differential on A , set $\omega = \sum_{i=1}^n a_i \omega_i$, with $a_i \in k(A)$; for $P \in A$, $\omega = \sigma_P \omega = \sum_i (\sigma_P a_i) \omega_i$, or $\sigma_P a_i = a_i$, or $a_i \in k$. Hence the invariant differentials on A form a k -module of order n . If ω is one of them, and U is a pole of ω , $\sigma_P U$ is also a pole of ω for each $P \in A$. Hence ω has infinitely many poles, a contradiction to 1.4. Therefore ω is of the first kind; in order to show that it is closed, it is sufficient to prove that each ω_i is closed; this will be a consequence of 1.7 if we can prove that $D_i D_j = D_j D_i$ ($i, j = 1, \dots, n$).

For sake of brevity, we shall make use of the functional notation of analysis, in spite of its ambiguous meaning. Let $\{x_1, \dots, x_m\}$ be a n.h.g.p. of A such that $x_i = 0$ at E_A (identity), and that $\{x_1, \dots, x_n\}$ form a regular set of parameters of $Q(E_A/A)$. We shall consider an algebraically closed extension K of k , of sufficiently high transcendency over k , and shall say that a point P of A_K , at finite distance for $\{x\}$, is general for A , if the values $y_1, \dots, y_m \in K$ of, respectively, $x_1, \dots, x_m$, at P are such that $k(y)$ is isomorphic to $k(x)$ over k , in an isomorphism in which y_i corresponds to x_i ($i = 1, \dots, m$). There are rational functions $g_1, \dots, g_m$ of two sets of indeterminates $\{X_1, \dots, X_m\}, \{Y_1, \dots, Y_m\}$, such that, for generic points P, Q of A , the co-ordinates of PQ are obtained by replacing, in $g_1, \dots, g_m$, the co-ordinates of P for $\{X\}$, and of Q for $\{Y\}$. Then §5 of [4] states that $D_i x_j = [\partial g_j(x, t) / \partial t_i]_{t=0}$, if $\{t\}$ are the co-ordinates of a point of A_K , general for A , and such that $k(x, t)$ has transcendency $2n$ over k . The previous formula is valid for $i = 1, \dots, n$ and $j = 1, \dots, m$; for the purpose of computing the partial derivatives, $t_{n+1}, \dots, t_n$ must be considered as functions of $t_1, \dots, t_n$. Then

$$\begin{aligned} D_h D_i x_j &= D_h [\partial g_j(x, t) / \partial t_i]_{t=0} \\ &= \sum_{r=1}^n [\partial^2 g_j(x, t) / \partial t_i \partial x_r]_{t=0} D_h x_r \\ &= \sum_r [\partial^2 g_j(x, t) / \partial t_i \partial x_r]_{t=0} [\partial g_r(x, z) / \partial z_h]_{z=0} \\ &= \sum_r [\partial^2 g_j(g(x, z), t) / \partial t_i \partial g_r(x, z)]_{t=z=0} [\partial g_r(x, z) / \partial z_h]_{z=0} \\ &= [\partial^2 g_j(x, g(z, t)) / \partial t_i \partial z_h]_{t=z=0}; \end{aligned}$$

this expression remains unchanged after interchanging i and h ; therefore $D_h D_i = D_i D_h$, as claimed. This proves completely the first statement of the theorem.

Now, let ω be a differential of the first kind on A ; let D be any invariant derivation on A , and set $a = \omega D$. If V is any $(n-1)$ -dimensional irreducible subvariety of A , $Q(V/A)$ is the join of the $Q(P/A)$ for $P \in V$. By Lemma 5.1 of [4], $Dx \in Q(P/A)$ if $x \in Q(P/A)$; hence $Dx \in Q(V/A)$ if $x \in Q(V/A)$. If t is a regular parameter of $Q(V/A)$, the derivation tD has a zero at V . Since ω does not have a pole at V , we conclude that $ta = \omega(tD) \in \mathfrak{P}(V/A)$, so that $a \in Q(V/A)$. This having to be true for any V , it follows that $a \in k$. If then $\{D_i\}$, $\{\omega_i\}$ have the previous meaning, and $\omega D_i = a_i \in k$, we necessarily have $\omega = \sum a_i \omega_i \in \mathfrak{D}_1(A)$, as claimed. The existence of an absolute basis for $\mathfrak{D}_1(A)$ is an immediate consequence of this fact, and of 1.8, Q.E.D.

Before proceeding any further, we remark that the differentials on a curve, in the language of [6], can be identified with our differentials thanks to Theorem 6 of Chapter VI of [6]; also the definitions of "first kind" and "second kind" are equivalent (see §8, Chapter VI of [6]). And it is apparent that if C is an absolutely irreducible curve without singularities over k , $\mathfrak{D}_1(C)$ has an absolute basis, and a representative of an absolute basis of $\mathfrak{D}_2(C)/\mathfrak{D}_e(C)$ exists. $\mathfrak{D}_1(C)$ is a k -module of order $g = \text{genus of } C$, while $\mathfrak{D}_2(C)/\mathfrak{D}_e(C)$ is a k -module of order $2g$ if k has characteristic zero. In the latter case, the k -modules $\mathfrak{D}_2(C)/\mathfrak{D}_1(C) + \mathfrak{D}_e(C)$ and $\mathfrak{D}_1(C)$ are dual to each other, by Theorem 8 of Chapter VII of [6]; the duality operation is induced by the operation $j(\omega_2, \omega_1)$, for $\omega_i \in \mathfrak{D}_i(C)$.

2.2. LEMMA. *Let A be an n -dimensional nonsingular abelian variety over the algebraically closed field k ; then each semi-invariant differential on A is of the second kind. If k has characteristic zero, the following statement is also true: let ω be a closed differential on A , such that for any $(n-1)$ -dimensional irreducible subvariety V of A there exists an $f \in k(A)$ for which $\omega - d_A f$ is regular at V ; then ω is semi-invariant, hence of the second kind. In particular, every element of $\mathfrak{D}_2(A)$ is semi-invariant.*

Proof. Let ω be a semi-invariant differential on A , and let U be an irreducible subvariety of A . Since ω has finitely many poles by 1.4, for a suitable $P \in A$ the differential $\sigma_P \omega$ does not have a pole at U ; as $\sigma_P \omega \sim \omega$, we conclude that ω is of the second kind.

In order to prove the second statement, assume k to have characteristic zero, and denote by $\mathfrak{D}'$ the set of the closed differentials on A having the property described in the statement of the lemma. For any differential ω on A , let $p(\omega)$ be its polar variety, that is, the join of all its poles, and let $e(\omega)$ denote the intersection of all the $p(\omega + d_A f)$ when f ranges in $k(A)$; if ω is closed, we have $\omega \in \mathfrak{D}'$ if and only if $\dim e(\omega) < n-1$. If $P \in A$, we have $e(\sigma_P \omega) = \sigma_P e(\omega)$, so that $\sigma_P \omega \in \mathfrak{D}'$ if and only if $\omega \in \mathfrak{D}'$.

Since A is the homomorphic image of a jacobian variety, there exist a curve C on k , without singularities, and a rational mapping ϕ of C into A , such that, for each $P \in A$, no proper group-subvariety of A contains $\sigma_P \phi C$, ϕC denoting the curve on A on which ϕ operates.

We shall now consider copies A, A_1, A_2 of A , and assume $k(A_2) \subseteq k(A \times A_1)$ as prescribed by the rational mapping of $A \times A_1$ onto A_2 which gives the law of composition on A ; as usual, and unless stated otherwise, if ω is any entity related to A , ω_i shall denote the copy of ω similarly related to A_i . Let $\{x\}, \{y\}$ be n.h.g.p. of A, A_1 respectively, copies of each other; set $H = k(A_1), L = k(A), K = k(C)$; let X be the point of A_H at which the co-ordinate x_i assumes the value y_i , and let Z be the point of $(A_i)_L$ at which the co-ordinate y_i assumes the value x_i ; set also $Y = \phi\{C\} \in A_K$.

Quite in general, if ω is a closed differential on the product $U \times V$ of two irreducible varieties, ω can be written uniquely as the sum $\omega_U + \omega_V$ of two differentials on $U \times V$, such that $\omega_U D = 0$ for any derivation D of $k(U \times V)$ over $k(U)$, while ω_V is similarly related to V ; we shall say that ω_U is the U -component of ω , and shall use this notation throughout this proof; ω_U is not, generally, a closed differential. Under the same conditions, let ω_U' be the differential on $U_{k(V)}$ such that $\omega_U' D = \omega_U D = \omega D$ for any derivation D on $U_{k(V)}$; then ω_U' is closed, and will be called the $U_{k(V)}$ -component of ω .

We shall now consider a differential $\omega \in \mathcal{D}'$, and shall denote by Ω the extension of ω_2 on $A \times A_1$; we shall also write $\Omega = \Omega_A + \Omega_{A_1}$, these being the A -component and the A_1 -component of Ω ; let ω^* be the A_H -component of Ω . Then we have $\omega^* = \sigma_X^{-1} \omega_H$, so that $e(\omega^*) = \sigma_X^{-1} e(\omega_H)$; and of course, any component of $e(\omega_H)$ is a subvariety of the extension over H of a component of $e(\omega)$. Since $e(C_H \times \omega^*) = C_H \times e(\omega^*)$, the previous result implies that the intersection of ϕ_H and $e(C_H \times \omega^*)$, on $C_H \times A_H$, is empty. Since each point of ϕ_H is simple on $C_H \times A_H$ and on ϕ_H , 1.4 and 1.5 apply to this case, and yield that $\phi_H \cap (C_H \times \omega^*)$ is a differential of the second kind on ϕ_H ; as $k(\phi)$ can be identified with K , there exists a unique differential α^* on C_H , of the second kind, such that $(\alpha^* \times A_H) \cap \phi_H = \phi_H \cap (C_H \times \omega^*)$. Let $\{\alpha_1, \dots, \alpha_r\}$ be a representative of an absolute basis of $\mathfrak{D}_2(C)/\mathfrak{D}_e(C)$; then, for suitable elements $z_i \in H$, we have $\alpha^* \sim \sum_i z_i(\alpha_i)_H$; this means that there exists an $f \in k(C \times A_1)$ such that the C_H -component $d^* f$ of $d_{C \times A_1} f$ has the property $\alpha^* = \sum_i z_i(\alpha_i)_H + d^* f$. Denote by α_C the differential on $C \times A_1$ such that $\alpha_C D = \alpha^* D$ for any derivation D on C_H , while $\alpha_C D = 0$ for any derivation D on $(A_1)_K$; then

$$2.3 \quad (\alpha_C \times A) \cap (\phi \times A_1) = (\phi \times A_1) \cap (C \times \Omega_A),$$

and $\alpha_C = \sum_i z_i(\alpha_i \times A_1) + (d_{C \times A_1} f)_C$.

We want to prove that $(\phi \times A_1) \cap (C \times \Omega_A)$ has no pole of the type $\phi \times U_1$, with U an $(n-1)$ -dimensional irreducible subvariety of A . In fact, consider a pole of this type; then, for any $P \in U$, $\phi \times P_1$ is a subvariety of a pole of $(\phi \times A_1) \cap (C \times \Omega_A)$, hence, by 1.5, also of a pole of $C \times \Omega_A$; if V is any pole of

ω , $V' = \sigma_X^{-1} V_H$ is a pole of ω^* , and $V'' = D_{V', A_1}$ is a pole of Ω_A , so that $C \times V''$ is a pole of $C \times \Omega_A$. Conversely, each pole of $C \times \Omega_A$ is of this type, because it operates on the whole A_1 ; consequently, if $\phi \times P_1$ is a subvariety of a pole of $C \times \Omega_A$, $Y \times (P_1)_K$ will be a subvariety of a V''_K , where V'' is of the previous type; this means that Y and $(P_1)_K$ correspond to each other in the algebraic correspondence V''_K ; but $V''[P_1] = \sigma_P^{-1} V$ (see §1 of [4]; in the notation of that section, after exchanging G_1, G_2, G_3 with, respectively, A_1, A_2, A , V' would be denoted by Γ_V , and V'' by T_V), so that $Y \in (\sigma_P^{-1} V)_K$, or $\sigma_{P_K} Y \in V_K$; this, in turn, implies $\sigma_P \phi C \subseteq V_K$ for any $P \in U$, or $\sigma_Q U = V$ for each $Q \in \phi C$. But this is impossible, since it implies that, for a fixed $R \in \phi C$, $\sigma_R^{-1} \phi C$ belongs to the proper group-subvariety of A consisting of the $P \in A$ such that $\sigma_P U = U$. It is thus proved that no pole of $(\phi \times A_1) \cap (C \times \Omega_A)$ is of the type $\phi \times U_1$; hence, by 2.3, $(\alpha_C \times A) \cap (\phi \times A_1)$ has the same property.

Now, for any $(n-1)$ -dimensional irreducible subvariety U_1 of A_1 , let $\{\xi_1, \dots, \xi_n\}$ be a set of uniformizing parameters at U_1 on A_1 ; for a given element z of K but not of k , there exists an element $g \in k(C \times A_1)$ such that $\alpha_C = g(d_C z \times A_1) = g d_{C \times A_1} z$; hence, $(\alpha_C \times A) \cap (\phi \times A_1) = g(d_{\phi z} \times A_1) = g d_{\phi \times A_1} z$, where z is now considered as an element of $k(\phi)$ in the identification $k(\phi) = K$. A set of uniformizing parameters at $\phi \times U_1$ on $\phi \times A_1$ is $\{z, \xi_1, \dots, \xi_n\}$, so that, by 1.2, the fact that $\phi \times U_1$ is not a pole of $(\alpha_C \times A) \cap (\phi \times A_1)$ means that $C \times U_1$ is not a pole of g ; as a consequence, and for the same reason, $C \times U_1$ is not a pole of α_C , and this is valid for each U_1 . We contend that, as a consequence, $z_i \in k$ for each i . In fact, if it is not so, after setting $d^* f = f' d^* z$ for $f' \in k(C \times A_1)$, let U_1 be a pole of some z_i , and let $p \in \mathfrak{P}(U_1/A_1)$ be such that $p z_i, p f' \in Q(C \times U_1/C \times A_1)$, one of them at least not being in $\mathfrak{P}(C \times U_1/C \times A_1)$. Then

$$0 = p \alpha_C \cap (C \times U_1) = \sum_i (\pi(p z_i))(\alpha_i \times U_1) + (\pi(p f'))(d_{C \times U_1} z)_C,$$

where π denotes reduction of $Q(C \times U_1/C \times A_1) \bmod \mathfrak{P}(C \times U_1/C \times A_1)$; hence,

$$\sum (\pi(p z_i))(\alpha_i)_M + (\pi(p f')) d_{C_M} z = 0,$$

where $M = k(U_1)$. Since the $(\alpha_i)_M$ represent an M -independent basis for $\mathfrak{D}_2(C_M)/\mathfrak{D}_e(C_M)$, we conclude that $\pi(p z_i) = \pi(p f') = 0$, against the hypothesis. We have thus shown that $z_i \in k$ for each i , so that there exists an element $\alpha = \sum_i z_i \alpha_i$ of $\mathfrak{D}_2(C)$, and 2.3 can be written as

$$[\alpha \times A \times A_1 + d_{C \times A \times A_1} f - C \times \Omega]_{A \times C} \cap (\phi \times A_1) = 0.$$

This also means that the ϕ -component of

$$[\alpha \times A \times A_1 + d_{C \times A \times A_1} f - C \times \Omega] \cap (\phi \times A_1)$$

is zero. As this is a closed differential, it must necessarily be of the type $\phi \times \beta$, with β a closed differential on A_1 . The $(A_1)_K$ -component of the above differen-

tial is then $\beta_K = (d'f - \Omega_K) \cap (Y \times (A_1)_K)$, where $d' = d_{(A_1)_K}$, or also $\beta_K = (d'f - \Omega_{A_1}) \cap (Y \times (A_1)_K)$. For any invariant derivation D on A , denote by D_1 also the extension of D_1 over K , or over $k(C \times A)$; then D_1 is regular at $Y \times (A_1)_K$, and we have $[(d'f - \Omega_{A_1}) \cap (Y \times (A_1)_K)]D_1 = D_1f - \pi[(\sigma_z^{-1}(\omega_1)_L)D_1]$, where π now denotes the homomorphic mapping of $Q(Y \times (A_1)_K/A_K \times (A_1)_K)$ onto $K((A_1)_K)$ whose kernel is $\mathfrak{P}(Y \times (A_1)_K/A_K \times (A_1)_K)$; the last expression coincides with

$$D_1f - \pi[\sigma_z^{-1}((\omega_1)_L D_1)] = D_1f - \sigma_{Y_1}^{-1}(\omega_1 D_1) = D_1f - (\sigma_{Y_1}^{-1}(\omega_1)_K)D_1;$$

thus, $\beta_K = d'f - \sigma_{Y_1}^{-1}(\omega_1)_K$. Now, for any $P \in C$, the above reasoning can be repeated, and it gives $\beta \sim (\sigma_Q^{-1}\omega)_1$ if $Q = \phi[P]$. If S is a fixed point of ϕC , and $C' = \sigma_S^{-1}\phi C$, let $Q_1, \dots, Q_n \in C'$; then $\omega \sim \sigma_{Q_1}^{-1}\omega$, hence $\omega \sim \sigma_{Q_2}^{-1}\omega \sim \sigma_{Q_1 Q_2}^{-1}\omega$, etc., so that $\omega \sim \sigma_R^{-1}\omega$ for $R = Q_1 Q_2 \dots Q_n$; but as the Q_i range over C' , R ranges over A , so that $\omega \sim \sigma_R \omega$ for any $R \in A$, and ω is semi-invariant, Q.E.D.

2.4. LEMMA. *Let A be an n -dimensional nonsingular jacobian variety over the algebraically closed field k of characteristic zero; then $\mathfrak{D}_2(A)/\mathfrak{D}_e(A)$ is a k -module of order $2n$, and there exists a representative of an absolute basis of it.*

Proof⁽²⁾. Let A be the jacobian of a curve C over k , without singular points. Let $C_1, \dots, C_n$ be copies of C , and set $V = C_1 \times \dots \times C_n$. Then, by [9], $k(A)$ is the set of the elements of $k(V)$ which are invariant for all the elements of the group G of the automorphisms of $k(V)$ over k which permute the $k(C_i)$ in all possible manners. Since there exists a representative of an absolute basis of $\mathfrak{D}_2(C)/\mathfrak{D}_e(C)$, 1.9 implies that there exists a representative of an absolute basis of $\mathfrak{D}_2(V)/\mathfrak{D}_e(V)$, and also that $\mathfrak{D}_2(V)/\mathfrak{D}_e(V)$ is a k -module of order $2n^2$. Let $\omega \in \mathfrak{D}_2(A)$, and let ω' be its extension on V ; then ω' is a closed differential, invariant for each element of G ; we contend that ω' is of the second kind. Let W be a nonempty irreducible subvariety of V ; by Theorem 6 of [9], if ϕ is the rational mapping of V onto A generated by the embedding $k(A) \subseteq k(V)$, the variety U on which $\phi[W]$ operates is such that $Q(U/A) \subseteq Q(W/V)$. There exists a $\omega_1 \sim \omega$ regular at U , and the extension ω'_1 of ω_1 on V is regular at W by 1.2, and clearly $\omega'_1 \sim \omega'$. Hence ω' is of the second kind. Conversely, let ω' be a closed differential of the second kind on V , invariant for each element of G . If $x_1, \dots, x_n \in k(A)$, and $k(A)$ is a finite extension of $k(x)$, also $k(V)$ is a finite extension of $k(x)$; write $\omega' = \sum_i a_i d_V x_i$, $a_i \in k(V)$. Then each a_i is invariant for each element of G , and therefore $a_i \in k(A)$, and ω' is the extension on V of $\omega = \sum_i a_i d_A x_i$. We contend that ω has the following property: for any $(n-1)$ -dimensional irreducible subvariety W of A , there exists an $f \in k(A)$ such that $\omega - d_A f$ is regular at W . In order to prove the assertion, we consider the distinct components $W_1, W_2, \dots$ of

(²) (Added November 4, 1955). The author is indebted to the referee of Amer. J. Math. for pointing out an error in the proof which the author had originally submitted for this lemma.

the subvariety of V on which $\phi[W]$ operates; they are all $(n-1)$ -dimensional, and conjugate to each other under the transformations of G (see §3 of [4]). Since ω' is of the second kind, there exists an $f_1 \in k(V)$ such that $\omega' - d_V f_1$ is regular at W_1 ; moreover, the $Q(W_j/V)$ are disjoint valuation rings in the sense of Krull, and therefore, by the independence theorem, it is possible to select an $f' \in \bigcap_{j \neq 1} Q(W_j/V)$ such that $f_1 - f' \in Q(W_1/V)$. Let m be the number of times W_1 (hence any W_j) appears among the conjugates of W_1 , under the transformations $g_1, g_2, \dots$ of G , and set $f = m^{-1} \sum_j g_j f'$; then, since $g_j \omega' = \omega'$ for each j , $\omega' - d_V f$ is regular at each W_j ; but $f \in k(A)$, so that $\omega' - d_V f$ is the extension on V of $\omega - d_A f$; since $Q(W/A) = k(A) \cap Q(W_1/V)$, we conclude that $\omega - d_A f$ is regular at W , as claimed. Thus, by 2.2, $\omega \in \mathfrak{D}_2(A)$.

We have thus established a 1-1 correspondence $\omega \rightarrow \omega'$ between $\mathfrak{D}_2(A)$, and the k -module $\mathfrak{s}$ of the elements of $\mathfrak{D}_2(V)$ which are invariant under the transformations of G . The k -module $\mathfrak{s}$ consists, by 1.9, of all the differentials of the type

$$\sum_{i=1}^n C_1 \times \cdots \times C_{i-1} \times \omega_i^* \times C_{i+1} \times \cdots \times C_n + d_V f,$$

where $f \in k(A)$ and ω_i^* is the copy on C_i of a $\omega^* \in \mathfrak{D}_2(C)$; clearly, $\mathfrak{s}/\mathfrak{s} \cap \mathfrak{D}_e(V)$ has order $2n$, so that $\mathfrak{D}_2(A)/\mathfrak{D}_e(A)$ has order $2n$. As a consequence, there exists an absolute basis of it, Q.E.D.

REMARK. In the notation of the preceding proof, and for a $\omega \in \mathfrak{D}_2(A)$, let ω' be the extension of ω on V , and let $\omega^* \in \mathfrak{D}_2(C)$ be such that

$$\omega' \sim \sum_i C_1 \times \cdots \times C_{i-1} \times \omega_i^* \times C_{i+1} \times \cdots \times C_n;$$

then the mapping $\omega \rightarrow \omega^*$ induces an isomorphism between $\mathfrak{D}_2(A)/\mathfrak{D}_e(A)$ and $\mathfrak{D}_2(C)/\mathfrak{D}_e(C)$, in which $\mathfrak{D}_1(A)$ and $\mathfrak{D}_1(C)$ map into each other.

Let A be an n -dimensional nonsingular abelian variety over the algebraically closed field k of characteristic zero; let C be a curve without singularities over k , and let λ be a rational mapping of C into A . For any $\omega \in \mathfrak{D}_2(A)$, we shall consider the element $\lambda^*(\omega) \in \mathfrak{D}_2(C)$ such that $(\lambda^*(\omega) \times A) \cap \lambda = (C \times \omega) \cap \lambda$, if these expressions have a meaning. Then, λ^* induces a homomorphic mapping of $\mathfrak{D}_2(A)/\mathfrak{D}_e(A)$ into $\mathfrak{D}_2(C)/\mathfrak{D}_e(C)$, such that $\mathfrak{D}_1(A)$ maps into $\mathfrak{D}_1(C)$. If, in particular, A is the jacobian of C , and λ is a canonical mapping of C into A , $\lambda^*(\omega)$ is the differential on C denoted by ω^* in the preceding remark (but for an exact differential); accordingly, in this particular case, the mapping $\lambda^*: \omega \rightarrow \lambda^*(\omega)$ induces an isomorphism between $\mathfrak{D}_2(A)/\mathfrak{D}_e(A)$ and $\mathfrak{D}_2(C)/\mathfrak{D}_e(C)$, in which $\mathfrak{D}_1(A)$ and $\mathfrak{D}_1(C)$ map into each other.

In general, λ can be extended to a homomorphism α of the jacobian J of C into A ; in other words, $\lambda = \alpha\phi$, if ϕ is a canonical mapping of C into J . If, in particular, α maps J onto A , then $\lambda^*(\omega) = \phi^*(\omega')$, if ω' is the extension of ω on J , when $k(A) \subseteq k(J)$ as prescribed by α . Clearly, $\omega' \in \mathfrak{D}_2(J)$ if $\omega \in \mathfrak{D}_2(A)$; and of course $\omega' \in \mathfrak{D}_1(J)$ if $\omega \in \mathfrak{D}_1(A)$; also, if $\omega \in \mathfrak{D}_2(A)$ and $\omega' \in \mathfrak{D}_1(J)$, then

necessarily $\omega \in \mathfrak{D}_1(A)$, by 2.1. Consequently, if α maps J onto A , or, equivalently, if, for any $P \in A$, no proper abelian subvariety of A contains $\sigma_P \lambda C$, λ^* induces an isomorphic mapping of $\mathfrak{D}_2(A)/\mathfrak{D}_e(A)$ into $\mathfrak{D}_2(C)/\mathfrak{D}_e(C)$, in which $\mathfrak{D}_1(A)$ maps into $\mathfrak{D}_1(C)$, and in which no element of $\mathfrak{D}_2(A)/\mathfrak{D}_e(A)$, but not of $\mathfrak{D}_1(A)$, maps into an element of $\mathfrak{D}_1(C)$.

Now, for an arbitrary A , let λ, μ be rational mappings of C into A . We shall denote by $\lambda + \mu$ the rational mapping ν of C into A such that $\nu[C] = (\lambda[C])(\mu[C])$. We contend that $(\lambda + \mu)^* \sim \lambda^* + \mu^*$, in the sense that $(\lambda + \mu)^*(\omega) \sim \lambda^*(\omega) + \mu^*(\omega)$ for any $\omega \in \mathfrak{D}_2(A)$. In fact, let A_1, A_2, A_3 be copies of A , and let D be the rational mapping of $A_1 \times A_2$ onto A_3 which gives the law of composition on A . Let $\omega_1, \omega_2, \omega_3$ be the copies of ω , on, respectively, A_1, A_2, A_3 . We shall denote by λ_1, μ_2, ν_3 the copies of, respectively, $\lambda, \mu, \nu = \lambda + \mu$ which map C on, respectively, A_1, A_2, A_3 . Then $(\omega_1 \times A_2 \times A_3 + \omega_2 \times A_1 \times A_3 - \omega_3 \times A_1 \times A_2) \cap D \sim 0$, since ω is semi-invariant, by 2.2. Also, $\theta = (\lambda_1 \times A_2 \times A_3) \cap (\mu_2 \times A_1 \times A_3) \cap (\nu_3 \times A_1 \times A_2)$ is a curve on $D \times C$ which, if considered as an algebraic correspondence between C and D , operates on the whole C . As a consequence, we have:

$$\begin{aligned}
 & [(\lambda^*(\omega) + \mu^*(\omega) - \nu^*(\omega)) \times A_1 \times A_2 \times A_3] \cap \theta \\
 &= \{[(\lambda^*(\omega) \times A_1) \cap \lambda_1] \times A_2 \times A_3\} \cap \theta \\
 &\quad + \{[(\mu^*(\omega) \times A_2) \cap \mu_2] \times A_1 \times A_3\} \cap \theta \\
 &\quad - \{[(\nu^*(\omega) \times A_3) \cap \nu_3] \times A_1 \times A_2\} \cap \theta \\
 &= \{[(C \times \omega_1) \cap \lambda_1] \times A_2 \times A_3\} \cap \theta + \{[(C \times \omega_2) \cap \mu_2] \times A_1 \times A_3\} \cap \theta \\
 &\quad - \{[(C \times \omega_3) \cap \nu_3] \times A_1 \times A_2\} \cap \theta \\
 &= [C \times (\omega_1 \times A_2 \times A_3 + \omega_2 \times A_1 \times A_3 - \omega_3 \times A_1 \times A_2)] \cap \theta \\
 &= [C \times (\cdots)] \cap (C \times D) \cap \theta = \{C \times [(\cdots) \cap D]\} \cap \theta \sim 0.
 \end{aligned}$$

Consequently, since θ operates on the whole C , we conclude that $\lambda^*(\omega) + \mu^*(\omega) - \nu^*(\omega) \sim 0$, as claimed.

2.5. LEMMA. *Let A be an abelian subvariety of the nonsingular abelian variety B over the algebraically closed field k of characteristic zero; then $\mathfrak{D}_1(A)$ is the set of the $\omega \cap A$, when ω ranges in $\mathfrak{D}_1(B)$; also, $\mathfrak{D}_2(A)$ is the set of the $\omega \cap A$ when ω ranges over all the elements of $\mathfrak{D}_2(B)$ which do not have a pole at A .*

Proof. B is isogenous to $A \times C$, for a suitable nonsingular abelian variety C ; assume $k(A \times C) \subseteq k(B)$ as prescribed by a homomorphism α of B onto $A \times C$ of positive degree, such that $\alpha A = A$. For any $\omega \in \mathfrak{D}_2(A \times C)$, let ω' be its extension on B ; then the mapping $\omega \rightarrow \omega'$ establishes an isomorphism between $\mathfrak{D}_2(A \times C)/\mathfrak{D}_e(A \times C)$ and $\mathfrak{D}_2(B)/\mathfrak{D}_e(B)$, in which $\mathfrak{D}_1(A \times C)$ and $\mathfrak{D}_1(B)$ correspond to each other. Thus it is sufficient to prove the lemma when B is replaced by $A \times C$, and A by $A \times E_C$. But in this case the lemma is a consequence of 1.9, Q.E.D.

2.6. LEMMA. *Let α be a homomorphism of an abelian variety A over the algebraically closed field k , into an abelian variety B over k , and, for a prime l not equal to the characteristic of k , let M be an l -adic matrix related to α as in Theorem 14 of [9]. Then the rank of M is twice the dimension of αA .*

Proof. If $\dim \alpha A = n$, there is an n -dimensional abelian subvariety C of A such that $\alpha C = \alpha A$; we may replace A with an isogenous variety $C \times C'$ without affecting ranks and dimensions, by Proposition 13 of [9]. Accordingly, we shall assume $A = C \times C'$, and $\alpha(C \times E_{C'}) = \alpha A$, while $\alpha(E_C \times C') = E_{\alpha A}$. After a suitable selection of l -adic co-ordinates on $C \times C'$, M assumes the form $(M_1 | M_2)$, where M_1, M_2 are the l -adic matrices related to, respectively, $[\alpha; C \times E_{C'}, B]$ and $[\alpha; E_C \times C', B]$. But then, by Proposition 13 and by Corollary 3 to Theorem 33 of [9], M_1 has rank $2n$, and $M_2 = 0$, Q.E.D.

Now, let J be the jacobian of a curve C , without singularities, over the algebraically closed field k ; by Corollary 2 to Theorem 22 of [9], there is an isomorphism $D \rightarrow \lambda$ between the ring of the classes of algebraic correspondences D between C and a copy C' of C , and the ring $\mathfrak{Q}$ of the endomorphisms λ of J . The interchange of C with C' induces an involution $\lambda \rightarrow \lambda'$ in $\mathfrak{Q}$, which in turn induces an involution in the algebra $\mathfrak{A}$ of the endomorphisms of J (=smallest algebra containing $\mathfrak{Q}$). This involution will be called the *Rosati involution* (see [7]).

2.7. LEMMA. *With these notations, λ and λ' have the same characteristic polynomial; if l is a prime different from the characteristic of k , and if L, L' are the l -adic matrices related to λ, λ' respectively, then L and L' have the same rank and the same characteristic polynomial.*

Proof^(*). We shall denote by $\mathfrak{A}^*$ the algebra of the l -adic matrices related to the elements of $\mathfrak{A}$. Let B be an abelian variety isogenous to J ; by §54 of [9], if μ is a homomorphism of B onto J , the correspondence $\lambda \rightarrow \lambda_1 = \mu^{-1}\lambda\mu$ is an isomorphism between $\mathfrak{A}$ and the algebra $\mathfrak{B}$ of the endomorphisms of B ; if L, L_1, M are the l -adic matrices related to, respectively, λ, λ_1, μ , we have $L_1 = M^{-1}LM$, and this proves that the isomorphism $L \rightarrow L_1$ preserves the characteristic polynomial and the rank. Moreover, if $\lambda'_1 = \mu^{-1}\lambda'\mu$, the mapping $\lambda_1 \rightarrow \lambda'_1$ is an involution in $\mathfrak{B}$. Theorem 25 of [9] implies that $\lambda\lambda' = 0$ if and only if $\lambda = 0$; consequently, $\lambda'_1\lambda_1 = 0$ if and only if $\lambda_1 = 0$. Now, by Theorem 28 of [9], we can select B to be of the type $B_1 \times \cdots \times B_r$, where each B_i is the direct product of simple abelian varieties, isomorphic to each other, while no abelian subvariety of B_i (for $i = 1, \dots, r$) is isogenous to any abelian subvariety of B_j if $j \neq i$. If $\mathfrak{B}_i$ is the algebra of the endomorphisms of B_i , by Theorem 29 of [9] $\mathfrak{B}_i$ is a simple algebra, and $\mathfrak{B}$ is the direct sum of $\mathfrak{B}_1, \dots, \mathfrak{B}_r$; more precisely, this is so after identification of an endomorphism

(*) This result is also an immediate consequence of the last formula of §76 of [9]; see however the remark which follows this proof.

ν of B_i with the endomorphism ν' of B such that $\nu'[P_1 \times \cdots \times P_r] = E_{B_1} \times \cdots \times E_{B_{i-1}} \times \nu P_i \times E_{B_{i+1}} \times \cdots \times E_{B_r}$, for $P_j \in B_j$.

Let $\mathfrak{B}'_i$ be the transform of $\mathfrak{B}_i$ in the involution $'$ on $\mathfrak{B}$; then $\mathfrak{B}'_i$ must coincide with some $\mathfrak{B}_j$, since the decomposition of a semi-simple algebra as direct sum of simple algebras is unique. Should j be different from i , an element $\lambda_i \in \mathfrak{B}_i$ would have the property that $\lambda'_i \lambda_i [P_1 \times \cdots \times P_r] = \lambda'_i [E_{B_1} \times \cdots \times E_{B_{i-1}} \times \lambda_i P_i \times E_{B_{i+1}} \times \cdots \times E_{B_r}] = E_B$, or $\lambda'_i \lambda_i = 0$, or finally $\lambda_i = 0$. Consequently, $\mathfrak{B}'_i = \mathfrak{B}_i$, and the involution $'$ induces an involution in each $\mathfrak{B}_i$. Now, for a suitable choice of l -adic co-ordinates in B , the l -adic matrix L_1 related to $\lambda_1 = \sum_i \lambda_{1i}$ ($\lambda_{1i} \in \mathfrak{B}_i$) can be written in the form

$$\begin{pmatrix} L_{11} & & & \\ & L_{12} & & \\ & & \ddots & \\ & & & L_{1r} \end{pmatrix},$$

where L_{1i} is the l -adic matrix related to λ_{1i} . When λ_1 ranges in $\mathfrak{B}$, L_{1i} ranges in the algebra $\mathfrak{B}_i^*$ of the l -adic matrices related to the elements of $\mathfrak{B}_i$; the involution $'$ induces an involution in each $\mathfrak{B}_i^*$, and $\mathfrak{B}_i^*$ is a simple subring of the algebra $\mathfrak{M}_i$ of all the matrices, with l -adic elements, of the same order as L_{1i} . Therefore, by a result on algebras⁽⁴⁾, there exists an element $N \in \mathfrak{M}_i$ such that, for each $L_{1i} \in \mathfrak{B}_i^*$, $L'_{1i} = N^{-1}(L_{1i})_{-1}N$, where $_{-1}$ denotes transposition of matrices. Thus L'_{1i} and L_{1i} have the same characteristic polynomial, also the same rank; hence L_1 and L'_1 , or also L and L' , have the same characteristic polynomial and the same rank, Q.E.D.

REMARK. It may be noted that the preceding proof, and the preceding lemma, are valid not only for the Rosati involution on a jacobian variety, but also for any reciprocity $'$ onto itself of the algebra $\mathfrak{A}$ of the endomorphisms of an abelian variety, provided such reciprocity has the property that $\lambda'\lambda = 0$ implies $\lambda = 0$ for $\lambda \in \mathfrak{A}$.

2.8. THEOREM. *Let A be a nonsingular n -dimensional abelian variety over the algebraically closed field k . Then the semi-invariant differentials on A are all of the second kind. If, in addition, k has characteristic zero, $\mathfrak{D}_2(A)/\mathfrak{D}_e(A)$ is a k -module of order $2n$, and there exists a representative of an absolute basis of it; also, each element of $\mathfrak{D}_2(A)$ is a semi-invariant differential.*

Proof. The first and last statements are part of 2.2. The second statement is true, by 2.4, if A is a jacobian variety. There remains to be proved the second statement for an arbitrary A . Let then A be arbitrary, and k be of characteristic 0. It is known that A is the homomorphic image of a jacobian

⁽⁴⁾ The result which we have in mind is a slight modification of the theorem on the extension of isomorphisms in a normal simple algebra; as this modification does not seem to be explicitly stated anywhere, we state and prove it in the Appendix at the end of this paper.

variety; hence A is isogenous to an abelian subvariety of a jacobian variety; since the k -modules of the closed differentials of the second kind, modulo exact differentials, of isogenous abelian varieties are isomorphic, we may assume A to be a subvariety of the (nonsingular) jacobian J of a curve C without singular points. Let $g = \dim J$ be the genus of C . Let J_1 be a copy of J , and let A_1 be the copy of A which is a subvariety of J_1 ; let λ be a homomorphism of J onto A_1 , and let ϕ be a canonical mapping of C into J . Then $\lambda\phi$ is a rational mapping of C into A_1 , also into J_1 . Let C_1 be a copy of C , and let ϕ_1 be the copy of ϕ which maps C_1 into J_1 . According to Corollary 2 of Theorem 22 of [9], λ is related to a class of algebraic correspondences between C and C_1 ; let D be an element of this class, so that D is a virtual cycle on $C \times C_1$. For any $P \in C$, we have $\lambda\phi[P] = \pi_i e_i \phi_1[Q_i]$, if $\sum_i e_i Q_i = D\{P\}^*$; let K be a finite extension of $k(C)$, such that $(D\{C\})_K$ is a sum of (not necessarily distinct) rational points $\Delta_1, \Delta_2, \dots$ of $(C_1)_K$. Then $((\lambda\phi)\{C\})_K = \prod_i (\phi_1)_K\{\Delta_i\}$. Let B be a model of K over k , without singularities, and let μ be the rational mapping of B into J_1 such that $\mu\{B\} = ((\lambda\phi)\{C\})_K$. Then the previous formula can be written $\mu = \sum_i \phi_1 D_i$, where D_i is the rational mapping of B into C_1 such that $D_i\{B\} = \Delta_i$. Therefore, in the notation of the discussion preceding 2.5, $\mu^* = \sum_i (\phi_1 D_i)^*$; this means that for any $\omega \in \mathfrak{D}_2(J)$, and for its copy ω_1 in $\mathfrak{D}_2(J_1)$, we have $\mu^*(\omega_1) = \sum_i (\phi_1^*(\omega_1))^{(i)}$, where $^{(i)}$ denotes extension on B of a differential on C_1 , when $k(C_1)$ is considered as a subfield of K as prescribed by Δ_i . Now, assume $D = \sum_i f_i H_i$, where the H_i are distinct irreducible algebraic correspondences between C and C_1 ; since $\mu^*(\omega_1)$ is the extension on B of $(\lambda\phi)^*(\omega_1)$, the previous formula states that

$$2.9 \quad (\lambda\phi)^*(\omega_1) = \sum_i f_i T_{k(H_i)/k(C)}(\phi_1^*(\omega_1))^{(i)},$$

where: (1) for each i , $k(C)$ and $k(C_1)$ are considered as subfields of $k(H_i)$ as prescribed by H_i ; (2) the symbol $()^{(i)}$ denotes extension on H_i of a differential on C , or on C_1 ; (3) T denotes trace.

Formula 2.9 can be simplified after introducing a different notation: we shall denote by $\mathfrak{D}_i(D)$ (for $i = 1, 2, e$) the direct sum $\sum_j \mathfrak{D}_i(H_j)$; for $\Omega \in \mathfrak{D}_i(C)$ (or $\mathfrak{D}_i(C_1)$), Ω' shall denote the element $\sum_j \Omega^{(j)}$ of $\mathfrak{D}_i(D)$; for $\Omega \in \mathfrak{D}_i(D)$, $T_{D/C}\Omega$ is defined as $\sum_j f_j T_{k(H_j)/k(C)}\Omega_j$, if $\Omega = \sum_j \Omega_j$ with $\Omega_j \in \mathfrak{D}_i(H_j)$; a similar definition holds for $T_{D/C_1}\Omega$. Finally, for $\Omega \in \mathfrak{D}_i(C)$ (or $\mathfrak{D}_i(C_1)$), $D^*(\Omega)$ is defined to be $T_{D/C_1}\Omega'$ (or $T_{D/C}\Omega'$). Then 2.9 becomes

$$2.10 \quad (\lambda\phi)^*(\omega_1) = D^*(\phi_1^*(\omega_1)).$$

This coincides with the correspondence, described in [7], between differentials of the first kind on curves related by an algebraic correspondence. If ω_1 ranges in $\mathfrak{D}_1(J_1)$, $\phi_1^*(\omega_1)$ spans the whole $\mathfrak{D}_1(C_1)$, while

$$(\lambda\phi)^*(\omega_1) = (\lambda\phi)^*(\omega_1 \cap A_1);$$

since $\omega_1 \cap A_1$ ranges over the whole $\mathfrak{D}_1(A_1)$ by 2.5, and since no proper abelian subvariety of A_1 contains $\lambda\phi C$, $(\lambda\phi)^*(\omega_1)$ spans a submodule of $\mathfrak{D}_1(C)$ of dimension n ; thus, by 2.6:

2.11. *If Ω ranges in $\mathfrak{D}_1(C_1)$, $D^*(\Omega)$ ranges over a submodule of $\mathfrak{D}_1(C)$ whose dimension n is such that $2n$ is the rank of the l -adic matrix (for an arbitrary prime l) related to the endomorphism of the jacobian of C which represents the class of D .*

Assume now ω_1 to range over the set of the elements of $\mathfrak{D}_2(J_1)$ which are regular at A_1 ; let $m \leq 2g$ be the order of $\mathfrak{D}_2(A_1)/\mathfrak{D}_e(A_1)$. Then, again, $\phi_1^*(\omega_1)$ ranges over the whole $\mathfrak{D}_2(C_1)$, and $(\lambda\phi)^*(\omega_1)$ ranges over a submodule Z of $\mathfrak{D}_2(C)$ such that $Z/Z \cap \mathfrak{D}_e(C)$ has order m . Thus, Z is the k -module of the $D^*(\Omega)$ when Ω ranges over $\mathfrak{D}_2(C_1)$, and we have $m = \text{ord } Z/Z \cap \mathfrak{D}_e(C)$. Let V be the k -module of the Ω' when Ω ranges in $\mathfrak{D}_2(C_1)$, and let W be the k -module of the $\Omega \in \mathfrak{D}_2(D)$ such that $T_{D/C}\Omega \in \mathfrak{D}_e(C)$. Then, obviously,

$$\begin{aligned} 2.12 \quad m &= \text{ord } Z/Z \cap \mathfrak{D}_e(C) = \text{ord } V/V \cap \mathfrak{D}_e(D) \\ &\quad - \text{ord } V \cap W/V \cap \mathfrak{D}_e(D) = 2g - \text{ord } V \cap W/V \cap \mathfrak{D}_e(D). \end{aligned}$$

Now, let Z' , V' , W' be defined in the same manner as Z , V , W respectively, after interchanging C with C_1 , and set $Z_0 = Z' \cap \mathfrak{D}_1(C_1)$, $V_0 = V' \cap \mathfrak{D}_1(D)$, $W_0 = W' \cap \mathfrak{D}_1(D)$ = set of all $\Omega \in \mathfrak{D}_1(D)$ such that $T_{D/C_1}\Omega = 0$. Let λ' be the correspondent of λ in the Rosati involution on J . The set of all $D^*(\Omega)$, when Ω ranges in $\mathfrak{D}_1(C)$, is Z_0 ; accordingly, by 2.11, 2.7 and 2.6, Z_0 has order n ; on the other hand, $\text{ord } Z_0 = \text{ord } V_0 - \text{ord } V_0 \cap W_0$, so that

$$2.13 \quad \text{ord } V_0 \cap W_0 = \text{ord } V_0 - n = g - n.$$

If $\Omega \in \mathfrak{D}_1(D)$ and $\Psi \in \mathfrak{D}_2(D)$, and if $\Omega = \sum_i \Omega_i$, $\Psi = \sum_i \Psi_i$, with $\Omega_i \in \mathfrak{D}_1(H_i)$, $\Psi_i \in \mathfrak{D}_2(H_i)$, we shall define $j(\Psi, \Omega)$ to be $\sum_i f_i j(\Psi_i, \Omega_i)$. Hence, if $\Omega \in \mathfrak{D}_1(D)$ is such that $j(\Psi, \Omega) = 0$ for each $\Psi \in V$, we also have $j(T_{D/C_1}\Psi, T_{D/C_1}\Omega) = 0$ for each $\Psi \in V$, or $j(\Psi, T_{D/C_1}\Omega) = 0$ for each $\Psi \in \mathfrak{D}_1(C_1)$; therefore, $T_{D/C_1}\Omega = 0$, or $\Omega \in W_0$. Conversely, for $\Omega \in \mathfrak{D}_1(D)$, $\Psi \in V$, and for any Ω^* conjugate to Ω over $k(C_1)$ (that is, if $\Omega = \sum_i \Omega_i$, $\Omega_i \in \mathfrak{D}_1(H_i)$, then $\Omega^* = \sum_i \Omega_i^*$, with Ω_i^* a conjugate of Ω_i on $k(C_1)$), we have $j(\Psi, \Omega) = j(\Psi, \Omega^*)$; hence $j(\Psi, \Omega) = 0$ if $\Omega \in W_0$. This proves that W_0 is the set of all the $\Omega \in \mathfrak{D}_1(D)$ such that $j(\Psi, \Omega) = 0$ for each $\Psi \in V$. In like manner we can prove that $W + \mathfrak{D}_1(D)$ is the set of the $\Psi \in \mathfrak{D}_2(D)$ such that $j(\Psi, \Omega) = 0$ for each $\Omega \in V_0$; consequently, V_0 is the set of the $\Omega \in \mathfrak{D}_1(D)$ such that $j(\Psi, \Omega) = 0$ for each $\Psi \in W$.

If g_i is the genus of H_i , set $\gamma = \sum_i g_i$; then, in view of the duality between $\mathfrak{D}_1(D)$ and $\mathfrak{D}_2(D)/\mathfrak{D}_1(D) + \mathfrak{D}_e(D)$, we have proved that

$$\begin{aligned} &\text{ord } V \cap W/V \cap W \cap [\mathfrak{D}_1(D) + \mathfrak{D}_e(D)] \\ &= \gamma - \text{ord } V_0 + W_0 = \gamma - (\text{ord } V_0 + \text{ord } W_0 - \text{ord } V_0 \cap W_0) \\ &= \gamma - g - \text{ord } W_0 + \text{ord } V_0 \cap W_0 = \gamma - n - \text{ord } W_0 \quad \text{by 2.13.} \end{aligned}$$

Now, the kernel of the homomorphic mapping $\Omega \rightarrow T_{D/C_1}\Omega$, of $\mathfrak{D}_1(D)$ onto $\mathfrak{D}_1(C_1)$, is W_0 , so that $\text{ord } W_0 = \gamma - g$, and

$$2.14 \quad \text{ord } V \cap W/V \cap W \cap [\mathfrak{D}_1(D) + \mathfrak{D}_e(D)] = g - n.$$

But $W \cap [\mathfrak{D}_1(D) + \mathfrak{D}_e(D)] = [W \cap \mathfrak{D}_1(D)] + \mathfrak{D}_e(D)$, so that

$$\begin{aligned} \text{ord } V \cap W/V \cap W \cap [\mathfrak{D}_1(D) + \mathfrak{D}_e(D)] \\ &= \text{ord } V \cap W/V \cap [(W \cap \mathfrak{D}_1(D)) + \mathfrak{D}_e(D)] \\ &= \text{ord } [V \cap W/V \cap \mathfrak{D}_e(D)]/[V \cap ((W \cap \mathfrak{D}_1(D)) + \mathfrak{D}_e(D))/V \cap \mathfrak{D}_e(D)] \\ &= \text{ord } V \cap W/V \cap \mathfrak{D}_e(D) - \text{ord } V \cap W \cap \mathfrak{D}_1(D); \end{aligned}$$

the last term, by the formula analogous to 2.13, has the value $g - n$, while the first member, by 2.14, has the value $g - n$; hence $\text{ord } V \cap W/V \cap \mathfrak{D}_e(D) = 2(g - n)$. This, replaced in 2.12, gives $m = 2g - 2(g - n) = 2n$, Q.E.D.

3. Factor sets. The definitions of factor sets, and of constant factor sets, are given in §§3 and 4 of [4].

3.1. THEOREM. *Let A be a nonsingular n -dimensional abelian variety over the algebraically closed field k of characteristic zero; let V be a 1-dimensional vector variety over k ; denote by $\Gamma, \Gamma_0, \Gamma_e$ the groups of, respectively, the factor sets of A into V , the factor sets associate to the identity of Γ , and the constant factor sets in Γ . Denote by $\mathfrak{D}'_2, \mathfrak{D}'_e$ the k -modules of the elements of, respectively, $\mathfrak{D}_2(A), \mathfrak{D}_e(A)$ which are regular at the identity of A . Then Γ/Γ_e is isomorphic to $\mathfrak{D}'_2/\mathfrak{D}_1(A)$, and in this isomorphism Γ_0/Γ_e corresponds to $\mathfrak{D}'_e$.*

Proof. Let A_1, A_2 be copies of A , and let v be a n.h.g.p. of V such that the law of composition on V is given by $v_3 = v_1 + v_2$. Denote by ∞ the point of V at infinity for v . It is readily seen that the multiplicative notation for the law of composition on V can be extended to yield $P\infty = \infty$ if $P \in V - \infty$. The associativity and commutativity properties remain true when meaningful. If $\gamma \in \Gamma - \Gamma_e$, assume $k(V) \subseteq k(A_1 \times A_2)$ as prescribed by γ . Let A_3 be another copy of A , and assume $k(A_3) \subseteq k(A_1 \times A_2)$ as prescribed by the law of composition on A . Let W be the radical of the "denominator" of v on $A_1 \times A_2$; then $W = \gamma[\infty]$. The relation

$$3.2 \quad (\gamma[P_1 + Q_2R_2])(\gamma[Q_1 \times R_2]) = (\gamma[P_1Q_1 \times R_2])(\gamma[P_1 \times Q_2]),$$

which characterizes factor sets, is now valid when: (a) none of the point $P_1 \times Q_2R_2, Q_1 \times R_2, P_1Q_1 \times R_2, P_1 \times Q_2$ belongs to the fundamental locus C of γ on $A_1 \times A_2$, and (b) one factor at least on each side is not ∞ . Let H be a component of W , operating on the whole $A_1 \times A_2$; we contend that there is a component of W of the type $Y_1 \times A_2$, where Y is an $(n-1)$ -dimensional irreducible subvariety of A , such that $H = T'_Y$ (see §4 of [4] for the definition of T'_Y). For if it were not so, it would be possible to find points P, Q, R of A such that $P_1 \times Q_2 \in H - (C \cap H)$, while $P_1Q_1 \times R_2, P_1 \times Q_2R_2$ and $Q_1 \times R_2$ do not be-

long to W ; and this would contradict the previous formula. Conversely, let Y be an $(n-1)$ -dimensional irreducible subvariety of A such that $Y_1 \times A_2$ is a component of W , but assume T'_Y not to be a component of W . Then again it is possible to find points P, Q, R of A such that $P_1 Q_1 \times R_2 \in W - C$, while $P_1 \times Q_2, P_1 \times Q_2 R_2, Q_1 \times R_2$ do not belong to W . Hence, since $\gamma[P_1 \times Q_2]$ is symmetrical in P, Q , we conclude that there are distinct irreducible $(n-1)$ -dimensional subvarieties $W^1, \dots, W^r$ of A , none of which contains E_A , such that W is the join of all the $W^1_1 \times A_2, A_1 \times W^i_2, T_{W^i}$.

If $P \in A$, let σ_P, σ_{P_2} be the automorphisms of $k(A_1 \times A_2)$ over, respectively, $k(A_2)$ and $k(A_1)$, which induce $(\sigma_P)_1, (\sigma_P)_2$ in $k(A_1), k(A_2)$ respectively. If $P, Q \in A$, denote by $v(P_1, Q_2)$ the element of k to which v is congruent mod $\mathfrak{P}(P_1 \times Q_2 / A_1 \times A_2)$, if $v \in Q(P_1 \times Q_2 / A_1 \times A_2)$. Denote also by $v(P_1), v(P_2)$ the elements of, respectively, $k(A_2), k(A_1)$ to which v is congruent modulo, respectively, $\mathfrak{P}(P_1 \times A_2 / A_1 \times A_2), \mathfrak{P}(A_1 \times P_2 / A_1 \times A_2)$. Then formula 3.2 can be written

$$3.3 \quad \sigma_{Q_2}^{-1} v + v(Q_1) = \sigma_{Q_1}^{-1} v + v(Q_2),$$

valid for $Q \in W^1 \cup \dots \cup W^r$.

It has been proved in §7 of [4] that, as a consequence of 3.3, there exists a closed differential ω on A such that

$$3.4 \quad d_{A_1 \times A_2} v = \omega'_3 - \omega'_1 - \omega'_2,$$

where ω'_i is the extension on $A_1 \times A_2$ of the copy ω_i of ω on A_i . This tells that ω is semi-invariant, so that $\omega \in \mathfrak{D}'_2$ by 2.8. The ω which satisfies 3.4 is unique but for an additive invariant differential; by 2.1, this means that 3.4 establishes a homomorphism $\gamma \rightarrow \omega + \mathfrak{D}_1(A)$ of Γ into $\mathfrak{D}'_2 / \mathfrak{D}_1(A)$, if ω is taken in $\mathfrak{D}_1(A)$ when $\gamma \in \Gamma_c$. It is also clear that γ, γ' correspond to the same $\omega + \mathfrak{D}_1(A)$ if and only if the difference of the corresponding v, v' has a differential = 0, that is, if and only if $\gamma - \gamma' \in \Gamma_c$; hence 3.4 establishes an isomorphic mapping of Γ / Γ_c into $\mathfrak{D}'_2 / \mathfrak{D}_1(A)$. Finally, $\gamma \in \Gamma_0$ if and only if ω can be selected in $\mathfrak{D}'_2$.

We must now prove that this isomorphic mapping is onto $\mathfrak{D}'_2 / \mathfrak{D}_1(A)$; given $\omega \in \mathfrak{D}'_2$, we shall select any $\gamma \in \Gamma_c$ if $\omega \in \mathfrak{D}_1(A)$; if not, let $\{x\}, \{y\}, \{z\}$ be n.h.g.p. of A_1, A_2, A_3 respectively, copies of each other, and let X be the rational point of $(A_2)_{k(A_1)}$ at which $y_i = x_i$. By 2.8, we have $(\omega_2)_{k(A_1)} \sim \sigma_X^{-1}(\omega_2)_{k(A_1)}$; let $\{v^1_i, \dots, v^n_i\}$ ($i=1, 2, 3$) be k -bases for $\mathfrak{D}_1(A_i)$, copies of each other, and write $\omega_i = \sum_j a^j_i v^j_i$, with $a^j_i \in k(A_i)$. Then there exists a $t \in k(A_1 \times A_2)$ such that $\sum_j a^j_2 (v^j_2)_{k(A_1)} = \sum_j a^j_3 (v^j_3)_{k(A_1)} - \sum_j (D^j_2 t) (v^j_1)_{k(A_1)}$, where $\{D^j_2\}$ is the k -basis for the invariant derivations on $(A_2)_{k(A_1)}$ such that $D^j_2 (v^i_1)_{k(A_1)} = \delta_{ij}$. Set $v^0_3 = (\text{extension of } v^j_3 \text{ on } A_1 \times A_2) = v^j_1 \times A_2 + v^j_2 \times A_1$; then the previous formula can be written:

$$3.5 \quad \sum_j a^j_2 v^j_2 \times A_1 = \sum_j a^j_3 v^j_0 - d_{A_1 \times A_2} t - \sum_j b_j v^j_1 \times A_2,$$

for suitable $b_j \in k(A_1 \times A_2)$. This, in turn, gives, if D_1^j are the copies of D_2^j on $(A_1)_{k(A_2)}$:

$$\sum_j b_j (v_1^j)_{k(A_2)} = \sum_j a_3^j (v_1^j)_{k(A_2)} - \sum_j (D_1^j t) (v_1^j)_{k(A_2)};$$

this means that $\sum_j b_j (v_1^j)_{k(A_2)} \sim \sigma_Y^{-1}(\omega_1)_{k(A_2)}$, where Y is the rational point of $(A_1)_{k(A_2)}$ at which $x_i = y_i$. Since $\sigma_Y^{-1}(\omega_1)_{k(A_2)} \sim (\omega_1)_{k(A_2)}$, it follows that $\sum_j b_j (v_1^j)_{k(A_2)} \sim (\omega_1)_{k(A_2)}$. Considering that the left side, and the first two terms of the right side of 3.5, are closed differentials on $A_1 \times A_2$, we see that the same is true of $\sum_j b_j v_1^j \times A_2$, so that $D_2^i b_j = 0$ for each i, j , and therefore $b_j \in k(A_1)$. But then $\sum_j b_j v_1^j \sim \omega_1$, or $\sum_j b_j v_1^j = \omega_1 + d_{A_1} t'$, for $t' \in k(A_1)$. If $u = t + t'$, 3.5 now gives

$$3.6 \quad d_{A_1 \times A_2} u = \omega_3' - \omega_1' - \omega_2',$$

which is analogous to 3.4. Let γ be the rational mapping of $A_1 \times A_2$ into V generated by setting $v = u$; we contend that γ is a factor set. If it is, it is clearly unique but for an additive constant set, since u is determined but for an additive element of k .

Now, $\gamma[E_{A_1} \times E_{A_2}]$ is a point $\neq \infty$ of V , because $E_{A_1} \times E_{A_2}$ does not belong to a pole of $\omega_3' - \omega_1' - \omega_2'$. Moreover, the operation of interchanging A_1 with A_2 transforms u into $u + h$, with $h \in k$; application of the same operation again shows that $2h = 0$, or $h = 0$. Hence $\gamma[P_1 \times Q_2] = \gamma[Q_1 \times P_2]$ for generic $P, Q \in A$.

For a generic $P \in A$, 3.6 implies that $d_2 \sigma_{P_1}^{-1} u = \sigma_{P_2}^{-1} \omega_3' - \omega_2'$, where d_2 indicates construction of differential on $(A_2)_{k(A_1)}$; also, $d_2 \sigma_{P_2}^{-1} u = \sigma_{P_2}^{-1} \omega_3' - \sigma_{P_2}^{-1} \omega_2'$; hence $d_2(\sigma_{P_1}^{-1} u - \sigma_{P_2}^{-1} u) = \sigma_{P_2}^{-1} \omega_2' - \omega_2' = ((\sigma_P^{-1} \omega - \omega)_2)_{k(A_1)}$. But $(\sigma_P^{-1} \omega - \omega)_2 = d_{A_2} u(P_1)$ by 3.6. Hence there exists a $w \in k(A_1)$ such that $w + \sigma_{P_1}^{-1} u = \sigma_{P_2}^{-1} u + u(P_1)$. Since P is generic, the elements $(\sigma_{P_1}^{-1} u)(E_{A_2})$, $(\sigma_{P_2}^{-1} u)(E_{A_2})$, $(u(P_1))(E_{A_2})$ exist, and equal respectively $(\sigma_P^{-1})_1(u(E_{A_2}))$, $u(P_2)$, $u(P_1, E_{A_2})$. But $u(E_{A_2}) \in k$, and therefore it coincides with $u(P_1, E_{A_2})$. Hence $w = u(P_2)$, so that $\sigma_{P_1}^{-1} u + u(P_2) = \sigma_{P_2}^{-1} u + u(P_1)$. This is the same as 3.3, and is therefore equivalent to 3.2, Q.E.D.

From 3.1, 2.8, and the fact that, by 2.2, $\mathfrak{D}_2' / \mathfrak{D}_1(A) + \mathfrak{D}_e' \cong \mathfrak{D}_2(A) / \mathfrak{D}_1(A) + \mathfrak{D}_e(A)$, we obtain:

3.7. COROLLARY. *Notations as in 3.1. Then Γ / Γ_0 is a k -module isomorphic to $\mathfrak{D}_2(A) / \mathfrak{D}_1(A) + \mathfrak{D}_e(A)$, and has order $n = \dim A$.*

Appendix (Added November 15, 1955). In the proof of 2.7 (see footnote in that proof) we have made use of the following result:

THEOREM. *Let $\mathfrak{A}$ be a total matrix algebra over the field F ; let $\mathfrak{B}, \mathfrak{B}'$ be subrings of $\mathfrak{A}$, such that $\mathfrak{B} \cap F = \mathfrak{B}' \cap F$ is a perfect subfield f of F , and assume $\mathfrak{B}, \mathfrak{B}'$ to be simple algebras over f , isomorphic to each other in an isomorphism σ over f . Assume also that for any element y of the center of $\mathfrak{B}$, or of $\mathfrak{B}'$, the characteristic*

polynomial of the matrix y has coefficients in f . Then there exists an inner automorphism of $\mathfrak{A}$ which induces σ between $\mathfrak{B}$ and $\mathfrak{B}'$.

Proof. Let $\mathfrak{C}$, $\mathfrak{C}'$ be the centers of $\mathfrak{B}$, $\mathfrak{B}'$ respectively; since f is perfect, we have $\mathfrak{C} = f(\theta)$ for a suitable $\theta \in \mathfrak{C}$, and $\mathfrak{C}' = f(\theta')$ if $\theta' = \sigma\theta$. For an indeterminate x , let $g(x)$, $g_0(x)$, $c(x)$ be respectively the minimal polynomial of θ over f , the minimal polynomial of θ over F , and the characteristic polynomial of the matrix θ ; let $g'(x) = g(x)$, $g'_0(x)$, $c'(x)$ have the same respective offices for θ' . Then g_0 is a divisor of g , g is a divisor of c , and the zeros of g_0 (in some algebraic closure of F) are all and only the zeros of c ; since g is separable, and has therefore distinct zeros, this implies that $g = g_0$, and likewise $g = g'_0$; also, $c = g^r$ for some positive integer r , and likewise $c' = g'^r$. Thus the matrices θ and θ' have the same characteristic polynomial, and of course $F(\theta)$ and $F(\theta')$ are semisimple algebras. This suffices to show that there exists an inner automorphism ρ of $\mathfrak{A}$ such that $\rho\theta = \theta'$; thus, ρ induces σ between $\mathfrak{C}$ and $\mathfrak{C}'$.

The smallest sub-algebra $F(\theta)$ of $\mathfrak{A}$ which contains $\mathfrak{C}$ and F is a semi-field, that is, a direct sum of fields, say $\mathfrak{C}_1 \dot{+} \cdots \dot{+} \mathfrak{C}_s$; likewise, the smallest sub-algebra of $\mathfrak{A}$ which contains $\mathfrak{C}'$ and F is $\mathfrak{C}'_1 \dot{+} \cdots \dot{+} \mathfrak{C}'_s$, where $\mathfrak{C}'_i = \rho\mathfrak{C}_i$; since θ has the same degree over f or F , we have $\mathfrak{C}_1 \dot{+} \cdots \dot{+} \mathfrak{C}_s = \mathfrak{C}_F = \mathfrak{C} \times F$ (direct product over f). This proves that the smallest sub-algebra of $\mathfrak{A}$ which contains $\mathfrak{B}$ and F is $\mathfrak{B} \times F = \mathfrak{B}_F$, and it is known that $\mathfrak{B}_F = \mathfrak{B}_1 \dot{+} \cdots \dot{+} \mathfrak{B}_s$, $\mathfrak{B}_i$ being a simple algebra over F , with the center $\mathfrak{C}_i$; similarly, the smallest sub-algebra of $\mathfrak{A}$ which contains $\mathfrak{B}'$ and F is $\mathfrak{B}' \times F = \mathfrak{B}'_F = \mathfrak{B}'_1 \dot{+} \cdots \dot{+} \mathfrak{B}'_s$. But then, σ can be extended to an isomorphism of $\mathfrak{B}_F$ onto $\mathfrak{B}'_F$ over F , and we have of course $\sigma\mathfrak{B}_i = \mathfrak{B}'_i$. Let e_i be the unit element of $\mathfrak{B}_i$, and set $\mathfrak{A}_i = e_i\mathfrak{A}e_i$; then $\mathfrak{A}_i$ is a total matrix algebra over F , containing, as simple sub-algebras, $\mathfrak{B}_i$ and $\rho^{-1}\sigma\mathfrak{B}_i$, since $\sigma\mathfrak{B}_i$ has the unit element $\sigma e_i = \rho e_i$, and is therefore a sub-algebra of $(\rho e_i)\mathfrak{A}(\rho e_i) = \rho\mathfrak{A}_i$. Thus, by the theorem on the extension of isomorphisms between simple sub-algebras of a normal simple algebra, there is an inner automorphism

$$\tau_i: y \rightarrow t_i y t_i^{-1}$$

of $\mathfrak{A}_i$ which induces $\rho^{-1}\sigma$ between $\mathfrak{B}_i$ and $\rho^{-1}\sigma\mathfrak{B}_i$. Set $t = t_1 + \cdots + t_s$, and let τ be the inner automorphism $y \rightarrow tyt^{-1}$ of $\mathfrak{A}$. For a $y \in \mathfrak{B}$, write $y = \sum_i y_i$, with $y_i \in \mathfrak{B}_i$; then $\sigma y = \sum_i \sigma y_i = \sum_i \rho \tau_i y_i = \rho \tau y$, Q.E.D.

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